Electronic Journal of Qualitative Theory of Differential Equations
2011, No. **68**, 1-21; **http://www.math.u-szeged.hu/ejqtde/**

First-order Three-Point BVPs at Resonance (II)

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Abstract:

This paper deals with existence of solutions to three-point BVPs in perturbed systems of first-order ordinary differential equations at resonance. An existence theorem is established by using the Theorem of Borsuk and some examples are given to illustrate it. A result for computing the local degree of polynomials whose terms of highest order have no common real linear factors is also presented.

Keywords: Three-point Boundary Value Problems, Theorem of Borsuk, Resonance Case.

2000 Mathematics Subject Classification: 34B10

1 Introduction

In this paper, we consider

$$x' - A(t)x = H(t, x, \varepsilon) = \varepsilon F(t, x, \varepsilon) + E(t), \ 0 \leq t \leq 1, \tag{1}$$

$$Mx(0) + Nx(\eta) + Rx(1) = 0, \tag{2}$$

where M, N and R are constant square matrices of order n, $A(t)$ is an $n \times n$ matrix with continuous entries, $E : [0, 1] \to \mathbb{R}$ continuous, $F : [0, 1] \times \mathbb{R}^n \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ is a continuous function and $\varepsilon \in \mathbb{R}$ such that $\mid \varepsilon \mid < \varepsilon_0$, and $\eta \in (0, 1)$.

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The work is motivated by Cronin [6, 7] who considered the problem of finding periodic solutions of perturbed systems. We adapt her approach to study three-point BVPs with linear boundary conditions using the methods and results of Cronin [6, 7]. The three-point BVP (1), (2) is called resonant or degenerate in the case that the rank of matrix $\mathcal{L} = n - r$, $0 < n - r < n$, that is the matrix $\mathcal{L} = M + NY_0(\eta) + RY_0(1)$ is singular where M, N and R are the constant $n \times n$ matrices given in (1), and $Y(t)$ is a fundamental matrix of linear system $x' = A(t)x$ and $Y_0(t) = Y(t)Y^{-1}(0)$. In studying the resonant case, we will use a finite-dimensional version of the Lyapunov Schmidt procedure (see [7]).

The existence of solutions to two-point, three-point, four-point or multipoint BVPs for ODEs at resonance have been studied by a number of authors (see, for example [4], [9], [10], [12], [13], [14], [15], [16], [20], [21], [22], [23], [24], [40], [32]), [17], [18], [19], [28], [36], [39], [41]). A great amount of work has been completed on the existence of solutions to BVPs for nonlinear systems of first-order ODEs at resonance which involve a small parameter (see, for example [5], [26], [27] and [37]). The resonance case for systems of first-order difference and differential equations has been considered by several authors (see for example Agarwal [1], Agarwal and O'Regan [2], Agarwal and Sambandham [3], Etheridge and Rodriguez [11], Rodriguez [33, 34, 35] and [38]). In these cases, resonance happens where the associated linear homogeneous BVP admits nontrivial solutions.

Recently, Mohamed et al. [30] established the existence of solutions at resonance for the following nonlinear boundary conditions

$$x' - A(t)x = H(t, x, \varepsilon) = \varepsilon F(t, x, \varepsilon) + E(t), \ 0 \le t \le 1, \tag{3}$$

$$Mx(0) + Nx(\eta) + Rx(1) = \ell + \varepsilon g(x(0), x(\eta), x(1)), \tag{4}$$

where M, N and R are constant square matrices of order n, $A(t)$ is an $n \times n$ matrix with continuous entries, $E : [0, 1] \to \mathbb{R}$ is continuous, $F : [0, 1] \times \mathbb{R}^n \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ is a continuous function where $\varepsilon_0 > 0$, $\ell \in \mathbb{R}^n$, $\eta \in (0, 1)$ and $g : \mathbb{R}^{3n} \to \mathbb{R}^n$ is continuous. They applied a version of Brouwer's fixed point theorem which is due to Miranda (see Piccinini, Stampacchia and Vidossich [31]) to prove the existence of solutions to (3), (4).

In this paper, we make use of the Theorem of Borsuk to show the existence of solutions of the BVP (1), (2) under suitable assumptions on the coefficients. We obtain the existence of solutions of three-point BVPs at resonance for general BVPs. We also present a result for computing the degree of $\psi_0(c) = (\psi_0^1(c_1, c_2), \psi_0^2(c_1, c_2))$ at $(0, 0)$ where the $\psi_0(c_1, c_2)$ are polynomials whose terms of highest order have no common real linear factors; see Cronin [7] p. 296-297. This result is for homogeneous polynomials in two variables which need not be odd functions while Borsuk's Theorem holds for continuous odd functions in any dimensions. These results generalize the degenerate case of periodic BVPs considered by Cronin [6, 7], and also the degenerate case of three-point BVP [13, 30].

2 Preliminaries

Lemma 2.1. *Consider the system*

$$x' = A(t)x \tag{5}$$

where $A(t)$ is an $n \times n$ matrix with continuous entries on the interval [0,1]. Let $Y(t)$ be a fundamental matrix of (5). Then the solution of (5) which satisfies the initial condition

$$x(0) = c \tag{6}$$

is $x(t) = Y(t)Y^{-1}(0)c$ where c is a constant n-vector. Abbreviate $Y(t)Y^{-1}(0)$ to $Y_0(t)$. Thus $x(t) = Y_0(t)c$.

Lemma 2.2. *[30] Let $Y(t)$ be a fundamental matrix of (5). Then any solution of (1) and (6) can be written as*

$$x(t, c, \varepsilon) = Y_0(t)c + \int_0^t Y(t)Y^{-1}(s)H(s, x(s), \varepsilon)ds. \tag{7}$$

The solution (1) satisfies the boundary conditions (2) if and only if

$$\mathcal{L}c = \varepsilon\mathcal{N}(c, \alpha, \eta, \varepsilon) + d \tag{8}$$

where

$$\mathcal{L} = M + NY_0(\eta) + RY_0(1), \mathcal{N}(c,\alpha,\eta,\varepsilon) = -\Big(\int_0^\eta NY(\eta)Y^{-1}(s)F(s,x(s,c,\varepsilon),\varepsilon)ds$$

$$+\int_0^1 RY(1)Y^{-1}(s)F(s,x(s,c,\varepsilon),\varepsilon)ds - g(c,x(\eta),x(1))\Big),$$

$$d = -\Big(\int_0^\eta NY(\eta)Y^{-1}(s)E(s)ds + \int_0^1 RY(1)Y^{-1}(s)E(s)ds - \ell\Big),$$

and $x(t,c,\varepsilon)$ *is the solution of (1) given* $x(0) = c$*.*

Thus (8) is a system of n real equations in $\varepsilon, c_1, \cdots, c_n$ where $c_1, \cdots, c_n$ are the components of c. The system (8) is sometimes called the branching equations.

Next we suppose that $\mathcal{L}$ is a singular matrix. This is sometimes called the resonance case or degenerate case. Now we consider the case rank $\mathcal{L} = n - r$, $0 < n - r < n$. Let E_r denote the null space of $\mathcal{L}$ and let E_{n-r} denote the complement in $\mathbb{R}^n$ of E_r, i.e.

$$\mathbb{R}^n = E_{n-r} \oplus E_r (\text{direct sum}).$$

Let $x_1, \cdots, x_n$ be a basis for $\mathbb{R}^n$ such that $x_1, \cdots, x_r$ is a basis for E_r, and $x_{r+1}, \cdots, x_n$ a basis for E_{n-r}.

Let P_r be the matrix projection onto $Ker\ \mathcal{L} = E_r$, and $P_{n-r} = I - P_r$, where I is the identity matrix. Thus P_{n-r} is a projection onto the complementary space E_{n-r} of E_r, and

$$P_r^2 = P_r,\ P_{n-r}^2 = P_{n-r} \text{ and } P_{n-r}P_r = P_rP_{n-r} = 0. \tag{9}$$

Without loss of generality, we may assume

$$P_r c = (c_1, \cdots, c_r, 0, \cdots, 0) \text{ and } P_{n-r}c = (0, \cdots, 0, c_{r+1}, \cdots, c_n). \tag{10}$$

We will identify $P_r c$ with $c^r = (c_1, \cdots, c_r)$ and $P_{n-r}c$ with $c^{n-r} = (c_{r+1}, \cdots, c_n)$ whenever it is convenient to do so.

Let H be a nonsingular $n \times n$ matrix satisfying

$$H\mathcal{L} = P_{n-r}. \tag{11}$$

Matrix H can be computed easily (see Cronin [7]). The nature of the solutions of the branching equations depends heavily on the rank of the matrix $\mathcal{L}$.

Lemma 2.3. *[30] The matrix $\mathcal{L}$ has rank $n-r$ if and only if the three-point BVP (5) and $Mx(0)+Nx(\eta)+Rx(1)=0$ has exactly r linearly independent solutions.*

Next we give a necessary and sufficient condition for the existence of solutions of $x(t,c,\varepsilon)$ of three-point BVPs for $\varepsilon>0$ such that the solution satisfies $x(0)=c$ where $c=c(\varepsilon)$ for suitable $c(\varepsilon)$.

We need to solve (8) for c when ε is sufficiently small. The problem of finding solutions to (1) and (2) is reduced to that of solving the branching equations (8) for c as function of ε for $|\varepsilon|<\varepsilon_0$. So consider (8) which is equivalent to

$$\mathcal{L}(P_r+P_{n-r})c=\varepsilon\mathcal{N}((P_r+P_{n-r})c,\alpha,\eta,\varepsilon)+d.$$

Multiplying (8) by the matrix H and using (11), we have

$$P_{n-r}c=\varepsilon H\mathcal{N}((P_r+P_{n-r})c,\alpha,\eta,\varepsilon)+Hd, \tag{12}$$

where

$$H\mathcal{N}((P_r+P_{n-r})c,\alpha,\eta,\varepsilon)=-H\Big(\int_0^\eta NY(\eta)Y^{-1}(s)F(s,x(s,c,\varepsilon),\varepsilon)ds$$
$$+\int_0^1 RY(1)Y^{-1}(s)F(s,x(s,c,\varepsilon),\varepsilon)ds-g(c,x(\eta),x(1))\Big)$$

and

$$Hd=-H\Big(\int_0^\eta NY(\eta)Y^{-1}(s)E(s)ds+\int_0^1 RY(1)Y^{-1}(s)E(s)ds-\ell\Big).$$

Since the matrix H is nonsingular, solving (8) for c is equivalent to solving (12) for c. The following theorem due to Cronin [6, 7] gives a necessary condition for the existence of solutions to the BVP (1) and (2).

Theorem 2.4. *A necessary condition that (12) can be solved for c, with $|\varepsilon|<\varepsilon_0$, for some $\varepsilon_0>0$ is $P_rHd=0$.*

Definition 2.5. [30] Let E_r denote the null space of $\mathcal{L}$ and let E_{n-r} denote the complement in $\mathbb{R}^n$ of E_r. Let P_r be the matrix projection onto $Ker\ \mathcal{L}=E_r$, and $P_{n-r}=I-P_r$, where I is the identity matrix. Thus P_{n-r} is a projection onto the complementary space E_{n-r} of

E_r. If E_{n-r} is properly contained in $\mathbb{R}^n$ then E_r is an $r-$dimensional vector space where $0 < r < n$. If $c = (c_1, \cdots, c_n)$, let $P_r c = c^r$ and $P_{n-r} = c^{n-r}$, then define a continuous mapping $\Phi_\varepsilon : \mathbb{R}^r \to \mathbb{R}^r$, given by

$$\Phi_\varepsilon(c_1, \cdots, c_r) = P_r H \mathcal{N}(c^r \oplus c^{n-r}(c^r, \varepsilon), \alpha, \eta, \varepsilon), \tag{13}$$

where $c^{n-r}(c^r, \varepsilon) = c^{n-r}$ is a differentiable function of c^r and ε, $P_r H \mathcal{N}$ is interpreted as $(H\mathcal{N}_1, \cdots, H\mathcal{N}_r)$. Similarly we will sometimes identify $P_{n-r}c$ and c^{n-r}. Setting $\varepsilon = 0$, we have

$$\Phi_0(c_1, \cdots, c_r) = P_r H \mathcal{N}(c^r \oplus P_{n-r} H d, \alpha, \eta, 0),$$

where $c^{n-r}(c^r, 0) = P_{n-r} H d$; note that from the context $c^{n-r}(c^r, 0) = P_{n-r} H d$ is interpreted as $c^{n-r}(c^r, 0) = (Hd_{r+1}, \cdots, Hd_n)$. If $E_r = \mathbb{R}^n$ and $P_r = I$, then $P_{n-r} = 0$. Since $P_{n-r} = 0$ it follows that the matrix H is the identity matrix. Thus define a continuous mapping $\Phi_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$, given by $\Phi_\varepsilon(c) = \mathcal{N}(c, \alpha, \eta, \varepsilon)$. Setting $\varepsilon = 0$, we have $\Phi_0(c) = \mathcal{N}(c, \alpha, \eta, 0)$.

3 Main Results

Now we state the well known Theorem of Borsuk (see, for example, Piccinini, Stampacchia and Vidossich [31] p. 211).

Theorem 3.1. *Let $B_k \subseteq \mathbb{R}^n$ be a bounded open set that is symmetrical with respect to the origin (that is $B_k = -B_k$) and contains the origin. If $\Phi_0 : \bar{B}_k \to \mathbb{R}^n$ is continuous and antipodal*

$$\Phi_0(c) = -\Phi_0(-c), \ (c \in \partial B_k)$$

and if $0 \notin \Phi_0(\partial B_k)$, then $d(\Phi_0, B_k, 0)$ is an odd number (and thus nonzero).

Next we introduce the computation of the topological degree of a mapping in Euclidean 2-space defined by homogeneous polynomials. The methods and notations described below come from Cronin [7, 8]. Let

$$\Phi_0^1(c_1, c_2) = C_1 \prod_{i=1}^{n} (c_1 - a_i c_2)^{p_i},$$
$$\Phi_0^2(c_1, c_2) = C_2 \prod_{j=1}^{m} (c_1 - b_j c_2)^{p_j},$$

where C_1, C_2 are constants. (We include the possibility that some $a_i = \infty$ or some $b_j = \infty$; equivalently, that the factor $y - a_i x$ is equal to $-x$ or the factor $y - b_j x$ is equal to $-x$). The topological degree is resolved by examining the changes of sign of $\Phi_0^1(c_1, c_2)$ and $\Phi_0^2(c_1, c_2)$ as c_1, c_2 varies over the boundary of the ball B_k with centre at the origin and arbitrary radius when computing the topological degree of (Φ_0^1, Φ_0^2). We may omit the following factors since none of them affect the degree of (Φ_0^1, Φ_0^2) on B_k at 0.

1. Factors $(c_1 - a_i c_2)$ and $(c_1 - b_j c_2)$ where a_i and b_j have complex conjugates in Φ_0^1, respectively, Φ_0^2.

2. Factors $(c_1 - a_i c_2)$ or $(c_1 - b_j c_2)$ which appear with even exponents where a_i and b_j are real.

3. Factors $(c_1 - a_i c_2)$ and $(c_1 - a_{i+1} c_2)$, if there exists a pair a_i, a_{i+1} $(i < i+1)$ such that no b_j lies between them (i.e., there is no b_j such that $a_i < b_j < a_{i+1}$). Similarly for pairs b_j, b_{j+1}.

4. Factors $(c_1 - a_r c_2)$ and $(c_1 - a_s c_2)$, if a_r and a_s are the smallest and largest of the array of numbers $a_1, \cdots, a_n, b_1, \cdots, b_m$. Similarly factors $(c_1 - b_r c_2)$ and $(c_1 - b_s c_2)$, if b_r and b_s are the smallest and largest of the array of numbers $a_1, \cdots, a_n, b_1, \cdots, b_m$.

If there are no remaining factors in Φ_0^1 or Φ_0^2, then the topological degree is zero. We now state the second main theorem in this paper (see Cronin [7] p. 38-40).

Theorem 3.2. *If we assume that the terms of highest degree of $\Phi_0^1(c_1, c_2)$ and $\Phi_0^2(c_1, c_2)$ are homogenous polynomials with no common real linear factors after reduction using the* *conditions 1, 2, 3, and 4 above, then*

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_p < b_p$$

or

$$b_1 < a_1 < b_2 < a_2 < \cdots < b_p < a_p$$

for some integer $p \leq min\{m,n\}$. *In the first case the degree is* p, *while in the second case the degree is* $-p$. *Hence*

$$d(\Phi_0, B_k, 0) \neq 0$$

for B_k, *a ball with centre at the origin and sufficiently large radius. Then for sufficiently small* ε, $\mid \varepsilon \mid < \varepsilon_0$

$$d(\Phi_\varepsilon, B_k, 0) = d(\Phi_0, B_k, 0) \neq 0.$$

Hence there is a solution $x(t,c,\varepsilon)$ *of the BVP (1), (2) with* $x(0,c,\varepsilon) = c$ *where* $c \in B_k \subset \mathbb{R}^2$ *and* $\mid \varepsilon \mid < \varepsilon_0$ *for some* $\varepsilon_0 > 0$.

Remark 3.3. In this paper, we find that an arbitrarily small change in $A(t)$ will affect the structure of the set of solutions, and the value of the local degree will depend on how the function $f(t,y,y',\varepsilon)$ is changed.

4 Applications and Examples

In this section, we apply our results from the previous section and we start by considering the degenerate case for $\alpha = \sqrt{2}$ in the interval $[0, 2\pi]$ with rank $\mathcal{L}_{(\alpha=\sqrt{2})} = 1 < 2$. Thus, we consider

$$y'' + y = \varepsilon f(t,y,y',\varepsilon), \quad t \in [0, 2\pi], \tag{14}$$

$$y(2\pi) = \alpha y(\eta), \;\; y'(0) = 0, \tag{15}$$

where $\eta = \pi/4$, $\alpha = \sqrt{2}$ and $f \in C([0,1] \times \mathbb{R}^2 \times (-\varepsilon_0, \varepsilon_0); \mathbb{R})$.

Then we study the totally degenerate case, rank $\mathcal{L}=0$ for general boundary conditions and give an example where Borsuk's Theorem or Theorem 3.2 applies. We consider

$$y''+16\pi^2 y=\varepsilon f(t,y,y',\varepsilon),\quad t\in[0,1], \tag{16}$$

$$2y(0)-y(1/2)-y(1)=0, \tag{17}$$

$$-y'(1/2)+y'(1)=0, \tag{18}$$

where $\eta=1/2\in(0,1)$, $f\in C([0,1]\times\mathbb{R}^2\times(-\varepsilon_0,\varepsilon_0);\mathbb{R})$.

We will use the following facts in solving the examples.

$$\int_0^{1/2}\sin^n 4\pi s\cos^m 4\pi s\ ds\neq 0,$$
$$\int_0^{1}\sin^n 4\pi s\cos^m 4\pi s\ ds\neq 0 \tag{19}$$

if and only if both n and m are even.

$$\int_0^{1}\sin^n 2\pi s\cos^m 2\pi s\ ds\neq 0 \tag{20}$$

if and only if both n and m are even.

Rank $\mathcal{L}_{(\alpha=\sqrt{2})}=1<2$, $\alpha=\sqrt{2}$ and $y'(0)=0$.

The BVP (14), (15) is equivalent to

$$\begin{pmatrix}x_1'\\x_2'\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}+\varepsilon\begin{pmatrix}0\\f(t,x_1,x_2,\varepsilon)\end{pmatrix}, \tag{21}$$

$$\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}x_1(0)\\x_2(0)\end{pmatrix}+\begin{pmatrix}-\sqrt{2}&0\\0&0\end{pmatrix}\begin{pmatrix}x_1(\pi/4)\\x_2(\pi/4)\end{pmatrix}+\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}x_1(2\pi)\\x_2(2\pi)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}, \tag{22}$$

where $x=\begin{pmatrix}x_1\\x_2\end{pmatrix}$, $A=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, $M=\begin{pmatrix}0&0\\0&1\end{pmatrix}$, $N=\begin{pmatrix}-\sqrt{2}&0\\0&0\end{pmatrix}$, $R=\begin{pmatrix}1&0\\0&0\end{pmatrix}$,
$F(t,x,\varepsilon)=\begin{pmatrix}0\\f(t,x_1,x_2,\varepsilon)\end{pmatrix}$. We obtain $Y(t)=\begin{pmatrix}\cos t&\sin t\\-\sin t&\cos t\end{pmatrix}$, $Y_0(t)=\begin{pmatrix}\cos t&\sin t\\-\sin t&\cos t\end{pmatrix}$,

$Y_0(2\pi) = \begin{pmatrix} \cos 2\pi & \sin 2\pi \\ -\sin 2\pi & \cos 2\pi \end{pmatrix}$, $Y_0(\pi/4) = \begin{pmatrix} \cos \pi/4 & \sin \pi/4 \\ -\sin \pi/4 & \cos \pi/4 \end{pmatrix}$ and

$Y(t)Y^{-1}(s) = e^{A(t-s)} = \begin{pmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{pmatrix}$. Then by Lemma 2.2, solving the problem (21), (22) is reduced to that of solving $\mathcal{L}_{(\alpha=\sqrt{2})}c = \varepsilon \mathcal{N}(c, \alpha, \eta, \varepsilon) + d$ for c. Thus we find $\mathcal{L}_{(\alpha=\sqrt{2})}$ and $\mathcal{N}(c, \alpha, \eta, \varepsilon)$.

$$
\begin{aligned}
\mathcal{L}_{(\alpha=\sqrt{2})} &= M + NY_0(\pi/4) + RY_0(2\pi) \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{N}(c, \alpha, \eta, \varepsilon) &= -\int_0^{\pi/4} \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} e^{A(\frac{\pi}{4}-s)} F(s, x(s, c, \varepsilon), \varepsilon)\, ds \\
&\quad - \int_0^{2\pi} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} e^{A(2\pi-s)} F(s, x(s, c, \varepsilon), \varepsilon)\, ds \\
&= (\mathcal{N}_1(c, \alpha, \eta, \varepsilon), 0);
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{N}_1(c, \alpha, \eta, \varepsilon) &= \int_0^{\pi/4} \sqrt{2} \sin(\pi/4 - s) f(s, x_1(s, c, \varepsilon), x_2(s, c, \varepsilon), \varepsilon) ds \\
&\quad - \int_0^{2\pi} \sin(2\pi - s) f(s, x_1(s, c, \varepsilon), x_2(s, c, \varepsilon), \varepsilon) ds,
\end{aligned}
$$

and $d = 0$. Thus we have rank $\mathcal{L}_{(\alpha=\sqrt{2})} = 1$. Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ be a basis for $Ker(\mathcal{L}_{\alpha=\sqrt{2}})$, and $Ker(\mathcal{L}_{\alpha=\sqrt{2}})$=Span e_1. Let P_1 be the matrix projection onto $Ker(\mathcal{L}_{\alpha=\sqrt{2}})$, $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. So $P_2 = I - P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Set $H = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ so that $H\mathcal{L}_{(\alpha=\sqrt{2})} = P_2$. Since $d = 0$, it follows that $P_1Hd=0$. Therefore a necessary condition of Theorem 2.4 is satisfied. Then we apply Theorem 3.2. In order to study Φ_0, we must first obtain $x(t,c,0)$, that is the solution of $x'=A(t)x$. By Lemma 2.1, $x'=A(t)x$ has a solution $x(t)$ with $x(0)=c=(c_1,0)^T$, where $x_2(0)=0=c_2$. Thus (14), (15) has a solution if $\varepsilon=0$ namely $x_1(t,c,0)=c_1\cos t, x_2(t,c,0)=-c_1\sin t$. We compute

$$\begin{aligned} P_1H\mathcal{N}(c,\alpha,\eta,\varepsilon) &= \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}\mathcal{N}_1(c,\alpha,\eta,\varepsilon)\\ 0\end{pmatrix}\\ &= \begin{pmatrix}\mathcal{N}_1(c,\alpha,\eta,\varepsilon)\\ 0\end{pmatrix}. \end{aligned}$$

Thus $\Phi_\varepsilon(c_1)=\mathcal{N}_1(c^1,\alpha,\eta,\varepsilon)$, where $P_2c=c^2=\binom{0}{c_2}$ and $P_1c=c^1=\binom{c_1}{0}$. Setting $\varepsilon=0$, we have $\Phi_0(c_1)=\mathcal{N}_1(c^1,\alpha,\eta,0)$, where $c^2(c^1,0)=P_2Hd=0$. In system (21), let $f(t,x_1,x_2,\varepsilon)=ax_1^3+bx_2$ so that $f\in C([0,2\pi]\times\mathbb{R}^2\times(-\varepsilon_0,\varepsilon_0);\mathbb{R})$. Thus $f(t,c_1\cos\ t,-c_1\sin\ t,0)=ac_1^3\ \cos^3 t-bc_1\sin t$. Using condition (20), and thus

$$\begin{aligned} \Phi_0(c_1) &= \int_0^{\pi/4}\sqrt{2}\sin(\pi/4-s)f(s,c_1\cos\ s,-c_1\sin s,0)ds\\ &\quad-\int_0^{2\pi}\sin(2\pi-s)f(s,c_1\cos s,-c_1\sin s,0)ds.\\ &= \int_0^{\pi/4}\sqrt{2}\sin(\pi/4-s)(ac_1^3\cos^3 s-bc_1\sin s)ds\\ &\quad-\int_0^{2\pi}\sin(2\pi-s)(ac_1^3\cos^3 s-bc_1\sin s)ds\\ &= \int_0^{\pi/4}\{ac_1^3\cos^4 s-bc_1\cos s\sin\ s-ac_1^3\sin s\cos^3 s+bc_1\sin^2 sds-bc_1\pi\}ds\\ &= ac_1^3(\frac{3\pi}{32}+\frac{1}{16})-bc_1(\frac{7\pi}{8}+\frac{1}{2}). \end{aligned}$$

Since $\Phi_0(c_1)$ is odd, the local degree is odd and therefore nonzero. Then for sufficiently large B_k and sufficiently small ε, $d(\Phi_\varepsilon,B_k,0)=d(\Phi_0,B_k,0)\neq 0$.

Next we apply Borsuk's Theorem in Example 1, and then Theorem 3.2 in Example 2 to find the local degree of a mapping in Euclidean 2-space defined by homogeneous polynomials.

Rank $\mathcal{L} = 0$.

The BVP (16), (17) and (18) is equivalent to

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -16\pi^2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \varepsilon \begin{pmatrix} 0 \\ f(t, x_1, x_2, \varepsilon) \end{pmatrix} \tag{23}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1(1/2) \\ x_2(1/2) \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{24}$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 \\ -16\pi^2 & 0 \end{pmatrix}$, $M = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $N = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $R = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.
$Y(t) = e^{At} = \begin{pmatrix} \cos 4\pi t & \sin 4\pi t/(4\pi) \\ -4\pi \sin 4\pi t & \cos 4\pi t \end{pmatrix}$, $Y^{-1}(t) = \begin{pmatrix} \cos 4\pi t & -\sin 4\pi t/(4\pi) \\ 4\pi \sin 4\pi t & \cos 4\pi t \end{pmatrix}$,
$Y_0(t) = Y(t)Y^{-1}(0) = \begin{pmatrix} \cos 4\pi t & \sin 4\pi t/(4\pi) \\ -4\pi \sin 4\pi t & \cos 4\pi t \end{pmatrix}$, $Y_0(1/2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and
$Y_0(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then by Lemma 2.2, the problem of solving (23), (24) is reduced to that of solving $\mathcal{L}c = \varepsilon \mathcal{N}(c, \alpha, \eta, \varepsilon) + d$ for c. Thus we find $\mathcal{L}$ and $\mathcal{N}(c, \alpha, \eta, \varepsilon)$.

$$\begin{aligned} \mathcal{L} &= M + NY_0(1/2) + RY_0(1) \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus we have rank $\mathcal{L} = 0$. Let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, is a basis for $Ker(\mathcal{L})$, and $Ker(\mathcal{L}) = Span(e_1, e_2)$. Let P_1 be the matrix projection onto $Ker(\mathcal{L})$, $P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So

$P_2 = I - P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Set $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so that $H\mathcal{L} = P_2$. We obtain

$$\begin{aligned}
\mathcal{N}(c,\alpha,\eta,\varepsilon) &= -\int_0^{1/2} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos 4\pi s & -\sin 4\pi s/(4\pi) \\ 4\pi \sin\ 4\pi s & \cos 4\pi s \end{pmatrix} \\
&\quad \times \begin{pmatrix} 0 \\ f(s, x_1(s,c,\varepsilon), x_2(s,c,\varepsilon), \varepsilon) \end{pmatrix} ds \\
&\quad - \int_0^{1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos 4\pi s & -\sin 4\pi s/(4\pi) \\ 4\pi \sin 4\pi s & \cos 4\pi s \end{pmatrix} \\
&\quad \times \begin{pmatrix} 0 \\ f(s, x_1(s,c,\varepsilon), x_2(s,c,\varepsilon), \varepsilon) \end{pmatrix} ds
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{1/2} \begin{pmatrix} -\sin\ 4\pi s/(4\pi) \\ \cos\ 4\pi s \end{pmatrix} f(s, x_1(s,c,\varepsilon), x_2(s,c,\varepsilon), \varepsilon)) ds \\
&\quad + \int_0^{1} \begin{pmatrix} -\sin\ 4\pi s/(4\pi) \\ -\cos\ 4\pi s \end{pmatrix} f(s, x_1(s,c,\varepsilon), x_2(s,c,\varepsilon), \varepsilon)) ds \\
&= \begin{pmatrix} \mathcal{N}_1(c,\alpha,\eta,\varepsilon) \\ \mathcal{N}_2(c,\alpha,\eta,\varepsilon) \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{N}_1(c,\alpha,\eta,\varepsilon) &= -\int_0^{1/2} \sin 4\pi s/(4\pi) f(s, x_1(s,c,\varepsilon), x_2(s,c,\varepsilon), \varepsilon)) ds \\
&\quad - \int_0^{1} \sin 4\pi s/(4\pi) f(s, x_1(s,c,\varepsilon), x_2(s,c,\varepsilon), \varepsilon)) ds,
\end{aligned}$$

$$\mathcal{N}_2(c,\alpha,\eta,\varepsilon) = -\int_{1/2}^{1} \cos 4\pi s f(s, x_1(s,c,\varepsilon), x_2(s,c,\varepsilon), \varepsilon)) ds,$$

and $d = 0$. Since $d = 0$, it follows that $P_1 H d = 0$. Therefore a necessary condition of Theorem 2.4 is satisfied. Then we apply Theorem 3.2. In order to study Φ_0, we must first obtain $x(t,c,0)$, that is the solution of $x' = A(t)x$. By Lemma 2.1, $x' = A(t)x$ has a solution $x(t)$ with $x(0) = c = (c_1, c_2)^T$. Thus (16), (17), (18) has a solution if $\varepsilon = 0$ namely $x_1(t, c, 0) = c_1 \cos 4\pi t + c_2 \sin 4\pi t/(4\pi), x_2(t, c, 0) = -4\pi c_1 \sin 4\pi t + c_2 \cos 4\pi t$. We compute

$$P_1 H \mathcal{N}(c, \alpha, \eta, \varepsilon) \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{N}_1(c, \alpha, \eta, \varepsilon) \\ \mathcal{N}_2(c, \alpha, \eta, \varepsilon) \end{pmatrix}.$$

Thus

$$\Phi_\varepsilon(c_1, c_2) = \begin{pmatrix} \mathcal{N}_1(c, \alpha, \eta, \varepsilon) \\ \mathcal{N}_2(c, \alpha, \eta, \varepsilon) \end{pmatrix}.$$

Setting $\varepsilon = 0$, we have

$$\Phi_0(c_1, c_2) = \begin{pmatrix} \mathcal{N}_1(c, \alpha, \eta, 0) \\ \mathcal{N}_2(c, \alpha, \eta, 0) \end{pmatrix}.$$

Now we state an example where the value of the local degree depends on the function $f(t, y, y', \varepsilon)$.

Example 1

In system (23), let $f(t, x_1, x_2, \varepsilon) = x_2^3$ so that $f \in C([0, 2\pi] \times \mathbb{R}^2 \times (-\varepsilon_0, \varepsilon_0); \mathbb{R})$. Then

$$f(t, c_1 \cos 4\pi t + c_2 \sin 4\pi t/(4\pi), -4\pi c_1 \sin 4\pi t + c_2 \cos 4\pi t, 0) = -64\pi^3 c_1^3 \sin^3 4\pi t$$
$$+48\pi^2 c_1^2 c_2 \sin^2 4\pi t \cos 4\pi t - 12\pi c_1 c_2^2 \sin 4\pi t \cos^2 4\pi t + c_2^3 \cos^3 4\pi t.$$

Using condition (19), we obtain

$$\begin{aligned}
& \Phi_0^1(c_1, c_2) \\
= & -\int_0^{1/2} \{\frac{\sin 4\pi s}{4\pi} f(s, c_1 \cos 4\pi s + c_2 \sin 4\pi s/(4\pi), -4\pi c_1 \sin 4\pi s + c_2 \cos\ 4\pi s, 0)\} ds \\
& -\int_0^{1} \{\frac{\sin 4\pi s}{4\pi} f(s, c_1 \cos 4\pi s + c_2 \sin 4\pi s/(4\pi), -4\pi c_1 \sin 4\pi s + c_2 \cos\ 4\pi s, 0)\} ds \\
= & -\int_0^{1/2} \{16\pi^2 c_1^3 \sin^4 4\pi s + 3c_1 c_2^2 \sin^2 4\pi s \cos^2 4\pi s\} ds \\
& -\int_0^{1} \{16\pi^2 c_1^3 \sin^4 4\pi s + 3c_1 c_2^2 \sin^2 4\pi s \cos^2 4\pi s\} ds \\
= & \; 9\pi^2 c_1^3 + \frac{9c_1 c_2^2}{16}
\end{aligned}$$

and

$$\begin{aligned}
&\Phi_0^2(c_1,c_2)\\
= &-\int_{1/2}^{1}\cos 4\pi s\{f(s,c_1\cos 4\pi s+c_2\sin 4\pi s/(4\pi),-4\pi c_1\sin 4\pi s+c_2\cos\ 4\pi s,0)\}ds\\
= &-\int_0^{1/2}\{48\pi^2c_1^2c_2\sin^2 4\pi s\cos^2 4\pi s+c_2^2\cos^4 4\pi s\}ds\\
= &-3\pi^2c_1^2c_2+\frac{3c_2^3}{16\pi}.
\end{aligned}$$

Since $\Phi_0(c_1,c_2)=(\Phi_0^1(c_1,c_2),\Phi_0^2(c_1,c_2))$ is continuous, odd on ∂B_k and $0\notin\Phi_0(\partial B_k)$, the local degree is odd and therefore nonzero. Then for sufficiently large B_k and sufficiently small ε, $d(\Phi_\varepsilon,B_k,0)=d(\Phi_0,B_k,0)\neq 0$.

Example 2

In system (23), let $f(t,x_1,x_2,\varepsilon)=x_1^2\cos 4\pi t+x_2\cos^2 4\pi t+x_1\sin^2 4\pi t$ so that $f\in C([0,2\pi]\times\mathbb{R}^2\times(-\varepsilon_0,\varepsilon_0);\mathbb{R})$. Then

$$\begin{aligned}
f(t,c_1\cos\ 4\pi t+c_2\sin 4\pi t/(4\pi),-4\pi c_1\sin 4\pi t+c_2\cos 4\pi t,0)=c_1^2\cos^3 4\pi t\\
+\frac{c_1c_2}{2\pi}\cos^2 4\pi t\sin 4\pi t+\frac{c_2^2\cos 4\pi t\sin^2 4\pi t}{16\pi^2}-4\pi c_1\cos^2 4\pi t sin 4\pi t\\
+c_2\cos^3 4\pi t+c_1\sin^2 4\pi t\cos 4\pi t+\frac{c_2}{4\pi}\sin^3 4\pi t.
\end{aligned}$$

Using condition (19), we obtain

$$\begin{aligned}
&\Phi_0^1(c_1,c_2)\\
= &-\int_0^{1/2}\{\frac{\sin 4\pi s}{4\pi}f(s,c_1\cos 4\pi s+c_2\sin 4\pi s/(4\pi),-4\pi c_1\sin 4\pi s+c_2\cos\ 4\pi s,0)\}ds\\
&-\int_0^{1}\{\frac{\sin 4\pi s}{4\pi}f(s,c_1\cos 4\pi s+c_2\sin 4\pi s/(4\pi),-4\pi c_1\sin 4\pi s+c_2\cos\ 4\pi s,0)\}ds\\
= &-\int_0^{1/2}\{[\frac{c_1c_2}{8\pi^2}-c_1]\cos^2 4\pi s\sin^2 4\pi s+\frac{c_2\sin^4 4\pi s}{16\pi^2}\}ds\\
&-\int_0^{1}\{[\frac{c_1c_2}{8\pi^2}-c_1]\cos^2 4\pi s\sin^2 4\pi s+\frac{c_2\sin^4 4\pi s}{16\pi^2}\}ds\\
= &\frac{-3c_1c_2}{128\pi^2}+\frac{3c_1}{16}-\frac{9c_2}{64\pi^2}
\end{aligned}$$

and

$$
\begin{aligned}
& \Phi_0^2(c_1, c_2) \\
= & -\int_{1/2}^{1} \{\cos 4\pi s f(s, c_1 \cos 4\pi s + c_2 \sin 4\pi s/(4\pi), -4\pi c_1 \sin 4\pi s + c_2 \cos\ 4\pi s, 0)\} ds \\
= & -\int_{0}^{1/2} \{(c_1^2 + c_2)\cos^4 4\pi s + (c_2^2 + c_1)\cos^2 4\pi s \sin^2 4\pi s\} ds \\
= & -(\frac{3\pi c_1^2}{4} + \frac{c_2^2}{256\pi^2}) - (\frac{3\pi c_2}{4} + \frac{c_1}{16}).
\end{aligned}
$$

Let

$$
\Phi_0^1(c_1, c_2) = p_1(c_1, c_2) + q_1(c_1, c_2)
$$

$$
\Phi_0^2(c_1, c_2) = p_2(c_1, c_2) + q_2(c_1, c_2)
$$

where

$$
p_1(c_1, c_2) = \tfrac{-3c_1c_2}{128\pi^2}, \quad q_1 = \tfrac{3c_1}{16} - \tfrac{9c_2}{64\pi^2},
$$

$$
p_2(c_1, c_2) = -(\tfrac{3\pi c_1^2}{4} + \tfrac{c_2^2}{256\pi^2}), \quad q_2 = -(\tfrac{3\pi c_2}{4} + \tfrac{c_1}{16}).
$$

Hence $p_1(c_1, c_2)$ is a polynomial homogeneous of degree $m = 2$ in c_1 and c_2, $p_2(c_1, c_2)$ is a polynomial homogeneous of degree $n = 2$ in c_1 and c_2, and $q_i(c_1, c_2)$ consists of the term $kc_1^{l_1(i)} c_2^{l_2(i)}$ where $l_1^{(i)} + l_2^{(i)} = 1 < \min(m, n) = 2$ for $i = 1, 2$. Thus we define ψ_0 to be the mapping defined by

$$
\psi_0(c_1, c_2) \to (p_1(c_1, c_2), p_2(c_1, c_2)).
$$

Since p_1 and p_2 have no common real linear factors, then $d(\psi_0, B_k, 0)$ is defined for B_k of arbitrary radius. After reduction using the conditions 1 and 4 in Theorem 3.2, ψ_0 is a constant. Hence $d(\psi_0, B_k, 0) = 0$. If the radius of B_k is sufficiently large then $d(\Phi_0, B_k, 0) = d(\psi_0, B_k, 0)$. Hence for sufficiently large B_k and sufficiently small ε, $d(\Phi_\varepsilon, B_k, 0) = 0$. Do the solutions exist? The answer is yes, $y \equiv 0$ for each $\varepsilon < \varepsilon_0$, in fact this is the only solution of the BVP (16), (17), (18). The equation $\Phi_0(c_1, c_2) = (0, 0)$ has just one solution $(c_1, c_2) = (0, 0)$. This implies $y(t) = x_1(t, c, 0) = c_1 \cos 4\pi t + c_2 \sin 4\pi t/(4\pi) \equiv 0$. Thus a necessary and sufficient condition for BVP (16), (17), (18) to have trivial solution is $f(t, 0, 0, \varepsilon) \equiv 0$ for $t \in [0, 2\pi]$, $\varepsilon < \varepsilon_0$.

5 Acknowledgements

The authors are grateful to the anonymous referee for his/her helpful suggestions and comments.

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(Received April 17, 2011)

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